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Relating the RNS and pure spinor formalisms for the superstring

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Abstract: Recently, the superstring was covariantly quantized using the BRST-like operator $Q = \oint \lambda^\alpha d_\alpha$ where λ^α is a pure spinor and d_α are the fermionic Green-Schwarz constraints. By performing a field redefinition and a similarity transformation, this BRST-like operator is mapped to the sum of the Ramond-Neveu-Schwarz BRST operator and η_0 ghost. This map is then used to relate physical vertex operators and tree amplitudes in the two formalisms. Furthermore, the map implies the existence of a b ghost in the pure spinor formalism which might be useful for loop amplitude computations.

Keywords: Superstrings and Heterotic Strings, BRST Symmetry.

Dedicated to my father on his 75th birthday

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Contents

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1. Introduction

Super-Poincaré covariant quantization of the ten-dimensional superstring is an important problem which has attracted many different approaches. Recently, a new approach [1] was proposed using a BRST-like operator $Q = \oint \lambda^\alpha d_\alpha$ where λ^α is a pure spinor worldsheet variable and d_α are the super-Poincaré covariant fermionic Green-Schwarz constraints. Unlike all previous approaches, this pure spinor approach was used to construct physical vertex operators and compute non-vanishing superstring tree amplitudes in a manifestly super-Poincaré covariant manner.

Because the BRST-like operator Q involves second-class constraints, its construction is non-conventional so the validity of the pure spinor formalism needs to be verified. Previous steps in this direction include showing that the cohomology of Q reproduces the light-cone Green-Schwarz (GS) spectrum [2] and that tree amplitudes involving external massless states (with up to four fermions) coincide with the Ramond-Neveu-Schwarz (RNS) computations [3].

In this paper, we shall provide further evidence for the validity of the pure spinor formalism by finding a field redefinition and similarity transformation which maps the BRST-like operator $Q = \oint \lambda^\alpha d_\alpha$ into $Q' = Q_{RNS} + \oint \eta$ where Q_{RNS} is the RNS BRST operator and η is the RNS variable coming from fermionization of the super-reparameterization (β, γ) ghosts [4]. This map will then be used to relate physical vertex operators and tree amplitudes in the two formalisms.

In a fixed picture, the cohomology of $Q' = Q_{RNS} + \oint \eta$ in the "large" RNS Hilbert space is the same as the cohomology of Q_{RNS} in the "small" RNS Hilbert space.[1] However, as will be shown, picture-changing is a gauge transformation if one uses Q' in the "large" Hilbert space. So although any physical state can be represented by vertex operators in different pictures in the cohomology of Q_{RNS}, all such vertex operators are equivalent in the cohomology of Q'.[2] This will be important since spacetime-supersymmetry transformations in the RNS formalism only close up to picture-changing.

As was shown in [6], there is a field redefinition from the ten-dimensional RNS variables $(x^m, \psi^m, b, c, \beta, \gamma)$ into GS-like variables which manifestly preserves six spacetime supersymmetries and a U(5) subgroup of the Wick-rotated SO(10) Lorentz group. The worldsheet variables in this GS-like description of the superstring consist of ten spacetime coordinates x^m for $m = 0$ to 9, six fermionic superspace coordinates (θ^+, θ^a) and their conjugate momenta (p_+, p_a) for $a = 1$ to 5, and two chiral bosons (s, t).

The next step in relating the RNS and pure spinor formalisms is to add a "topological" sector to the GS-like variables consisting of ten fermionic superspace coordinates and their conjugate momenta, (θ_{ab}, p^{ab}), as well as ten bosonic coordinates and their conjugate momenta, (u_{ab}, v^{ab}). In order that physical states are

[1] In the language of [4], the "large" and "small" RNS Hilbert spaces refer to the spaces with and without the ξ zero mode.

[2] The term "cohomology" will always refer to "absolute cohomology" as opposed to "relative cohomology", i.e. the BRST cohomology without requiring annihilation by the b_0 ghost. At non-zero momentum, the two cohomologies are expected to agree but there may be picture-changing subtleties at zero momentum as discussed in [5].

unaffected by this topological sector, the BRST operator Q' will be modified to $Q_{\mathrm{U}(5)} = Q' + \frac{1}{2}\oint u_{ab}p^{ab} + \cdots$. Because of the standard quartet mechanism, states in the cohomology of $Q_{\mathrm{U}(5)}$ will be independent of the $(\theta_{ab}, p^{ab}, u_{ab}, v^{ab})$ variables.

The final step in relating the two formalisms is to find a similarity transformation R such that $e^{-R}Q_{\mathrm{U}(5)}e^R = \oint \lambda^\alpha d_\alpha$ where (θ^a, θ^+) combines with θ_{ab} to form a covariant spinor θ^α for $\alpha = 1$ to 16, (p_a, p_+) combines with p^{ab} to form a covariant spinor p_α, and d_α is constructed from $(x^m, p_\alpha, \theta^\alpha)$ in the usual super-Poincaré covariant manner. Furthermore, the chiral bosons (s, t) of the U(5) formalism combine with the bosons (u_{ab}, v^{ab}) of the topological sector to form a pure spinor λ^α and its conjugate momentum where u_{ab} parameterizes the ten-dimensional complex coset SO(10)/ U(5).

So after using the field redefinition to write any physical RNS vertex operator U_{RNS} in terms of U(5) variables, one can construct a vertex operator U in the cohomology of $Q = \oint \lambda^\alpha d_\alpha$ by defining $U = e^{-R}U_{RNS}e^R$. Using this map from physical RNS vertex operators to physical pure spinor vertex operators, it is straightforward to show that the tree amplitudes in the two formalisms are identical. This justifies the rather unconventional normalization prescription of [1] for integrating over worldsheet zero modes in the pure spinor formalism.

The only subtlety in relating the two formalisms is that although the operators $Q_{\mathrm{U}(5)}$ and R are manifestly spacetime supersymmetric, they are not manifestly Lorentz invariant. So pure spinor vertex operators U obtained by this map do not necessarily transform covariantly under Lorentz transformations. Note that the similarity transformation guarantees only that U transforms covariantly up to a BRST-trivial operator. However, evidence will be given that, with one exception which will be discussed in the following paragraph, there is always a suitable gauge choice for U which transforms covariantly.

The one exception is the ghost-number -1 operator $\int d^2z\ \mu(z)\ b(z)$ where b is the RNS b ghost and $\mu(z)$ is a Beltrami differential. When mapped to the pure spinor formalism, there is no gauge choice for which this operator is super-Poincaré invariant. Nevertheless, it will be argued that the OPE of this operator with any physical operator of positive ghost number can be written in super-Poincaré covariant form. Furthermore, this operator can be used in the pure spinor formalism to construct integrated vertex operators from unintegrated operators, to relate string antifields and fields, and to define tree amplitudes in a worldsheet reparameterization invariant manner. It is hoped that this ghost-number -1 operator will also be useful for computing loop amplitudes using the pure spinor formalism.

Section 2 of this paper will review the pure spinor formalism of the superstring, and section 3 will review the RNS formalism using a modified definition of physical states in which picture changing is a gauge transformation. In section 4, a map will be found which takes the sum of the RNS BRST operator and η_0 ghost into the BRST-like operator in the pure spinor formalism. In section 5, this map will be used to relate physical vertex operators and tree amplitudes in the two formalisms. In section 6, the b ghost will be constructed in the pure spinor formalism and section 7 will conclude with open problems and speculations.

2. Review of the pure spinor formalism

2.1 Pure spinors

In this section, we shall review the relevant features of the pure spinor formalism of the superstring. Following the work of Siegel [7], the action in this formalism is constructed using a first-order action for the θ^α worldsheet variables where the fermionic conjugate momenta, p_α, are independent worldsheet variables. In addition to the worldsheet variables $(x^m, \theta^\alpha, p_\alpha)$ for $m = 0$ to 9 and $\alpha = 1$ to 16, the action also depends on a bosonic "ghost" variable λ^α satisfying the pure spinor condition

$$\lambda^\alpha \gamma^m_{\alpha\beta} \lambda^\beta = 0 \qquad \text{for } m = 0 \text{ to } 9 \tag{2.1}$$

where $\gamma^m_{\alpha\beta}$ and $\gamma^{m\ \alpha\beta}$ are 16×16 symmetric matrices which form the off-diagonal blocks of the 32×32 ten-dimensional Γ-matrices in the Weyl representation.

In worldsheet conformal gauge, the worldsheet action is

$$S = \int d^2 z \left[\frac{1}{2} \partial x^m \overline{\partial} x_m + p_\alpha \overline{\partial} \theta^\alpha \right] + S_\lambda \tag{2.2}$$

with the free-field OPE's

$$x^m(y) x^n(z) \to -\eta^{mn} \log |y - z|^2 \,, \qquad p_\alpha(y) \theta^\beta(z) \to (y-z)^{-1} \delta_\alpha^\beta \,, \tag{2.3}$$

where S_λ is the action for the pure spinor variable which will be described more explicitly in the following paragraphs. In the above action and in the rest of this paper, we shall ignore the right-moving degrees of freedom. The results of this paper are easily generalized to describe the heterotic, type-I, or type-II superstrings by choosing the right-moving sector appropriately.

Since λ^α is constrained by (2.1), it is convenient to solve this constraint when defining S_λ. A parameterization of λ^α which preserves a U(5) subgroup of (Wick-rotated) SO(10) is[3]

$$\lambda^+ = e^s \,, \qquad \lambda_{ab} = u_{ab} \,, \qquad \lambda^a = -\frac{1}{8} e^{-s} \epsilon^{abcde} u_{bc} u_{de} \tag{2.4}$$

where $a = 1$ to 5, and $u_{ab} = -u_{ba}$ are ten complex variables parameterizing the SO(10)/ U(5) coset,[4] s is a complex phase, and the SO(10) spinor λ^α has been writ-

[3] To simplify comparison with the U(5) formalism of [6], the u_{ab} variables defined here differ from those of [1] by a factor of e^s (which was called γ in [1]).

[4] If one does not Wick rotate, the u_{ab} variables of (2.4) parameterize the compact space $SO(9,1)/(U(4) \times \mathcal{R}^9)$ where U(4) is a subgroup of the transverse SO(8) rotations and $\mathcal{R}^9$ represents the nine light-like boosts generated by M^{+m} [8].

ten in terms of its irreducible U(5) components which transform as $(1_{\frac{5}{2}}, \overline{10}_{\frac{1}{2}}, 5_{-\frac{3}{2}})$ representations of $SU(5)_{U(1)}$.[5] The λ^α parameterization of (2.4) is possible whenever $\lambda^+ \neq 0$.

Using the above parameterization of λ^α, one can define

$$S_\lambda = \int d^2z \left[\overline{\partial} t \partial s - \frac{1}{2} v^{ab} \overline{\partial} u_{ab} \right] \tag{2.5}$$

where t and v^{ab} are the conjugate momenta to s and u_{ab} satisfying the OPE's

$$t(y)\; s(z) \to \log(y-z)\,, \qquad v^{ab}(y)\, u_{cd}(z) \to \delta_c^{[a} \delta_d^{b]} (y-z)^{-1}\,. \tag{2.6}$$

Note that the factor of 1/2 in the $v^{ab}\overline{\partial} u_{ab}$ term has been introduced to cancel the factor of 2 from $u_{ab} = -u_{ba}$. Also note that s and t are chiral bosons, so their contribution to (2.5) needs to be supplemented by a chirality constraint. Furthermore, the zero modes of s and t can only appear through the exponentials e^{ms+nt} for integers m and n.

One can construct SO(10) Lorentz currents N^{mn} out of these free variables as[6]

$$N = \frac{1}{\sqrt{5}} \left(\frac{1}{4} u_{ab} v^{ab} + \frac{5}{2} \partial t - \frac{5}{2} \partial s \right), \qquad N_a^b = u_{ac} v^{bc} - \frac{1}{5} \delta_a^b u_{cd} v^{cd}\,, \tag{2.7}$$

$$N^{ab} = e^s v^{ab}\,, \qquad N_{ab} = e^{-s} \left(2\partial u_{ab} - u_{ab} \partial t - 2 u_{ab} \partial s + u_{ac} u_{bd} v^{cd} - \frac{1}{2} u_{ab} u_{cd} v^{cd} \right)$$

where N^{mn} has been written in terms of its U(5) components $(N, N_a^b, N^{ab}, N_{ab})$ which transform as $(1_0, 24_0, 10_2, \overline{10}_{-2})$ representations of $SU(5)_{U(1)}$. The Lorentz currents of (2.7) can be checked to satisfy the OPE's

$$N^{mn}(y) \lambda^\alpha(z) \to \frac{1}{2} (\gamma^{mn})^\alpha{}_\beta \frac{\lambda^\beta(z)}{y-z}\,, \tag{2.8}$$

$$N^{kl}(y) N^{mn}(z) \to \frac{\eta^{m[l} N^{k]n}(z) - \eta^{n[l} N^{k]m}(z)}{y-z} - 3 \frac{\eta^{kn}\eta^{lm} - \eta^{km}\eta^{ln}}{(y-z)^2}\,. \tag{2.9}$$

So although S_λ is not manifestly Lorentz covariant, any OPE's of λ^α and N^{mn} which are computed using this action are manifestly covariant.

In terms of the free fields, the stress tensor is

$$T = -\frac{1}{2} \partial x^m \partial x_m - p_\alpha \partial \theta^\alpha + \frac{1}{2} v^{ab} \partial u_{ab} + \partial t \partial s + \partial^2 s \tag{2.10}$$

[5] A simple way to obtain these U(5) representations is to write an SO(10) spinor using $[\pm\pm\pm\pm\pm]$ notation where Weyl/anti-Weyl spinors have an odd/even number of + signs. The $1_{\frac{5}{2}}$ component of λ^α is the component with five + signs, the $\overline{10}_{\frac{1}{2}}$ component has three + signs, and the $5_{-\frac{3}{2}}$ component has one + sign.

[6] This construction can be obtained from [1] by replacing u_{ab} with $u_{ab}e^s$ as explained in footnote 3. Similar free-field constructions can be found in [9].

where the $\partial^2 s$ term is included so that the Lorentz currents of (2.7) are primary fields. This stress tensor has zero central charge and can be written in manifestly Lorentz invariant notation as [10]

$$T = -\frac{1}{2}\partial x^m \partial x_m - p_\alpha \partial\theta^\alpha + \frac{1}{10}N_{mn}N^{mn} - \frac{1}{2}(\partial h)^2 - 2\partial^2 h \tag{2.11}$$

where h is a Lorentz scalar defined in terms of the free fields by

$$\partial h = \frac{1}{4}u_{ab}v^{ab} + \frac{1}{2}\partial t + \frac{3}{2}\partial s\,. \tag{2.12}$$

Note that h has no singularities with N^{mn} and satisfies the OPE's

$$h(y)h(z) \to -\log(y-z)\,, \qquad \partial h(y)\lambda^\alpha(z) \to \frac{1}{2}(y-z)^{-1}\lambda^\alpha(z)\,.$$

The operator $2\oint \partial h$ will be identified with the ghost-number operator so that λ^α carries ghost number +1.

2.2 Physical states

Physical states in the pure spinor formalism are defined as super-Poincaré covariant ghost-number +1 states in the cohomology of the BRST-like operator

$$Q = \oint \lambda^\alpha d_\alpha \tag{2.13}$$

where

$$d_\alpha = p_\alpha - \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m - \frac{1}{8}\gamma^m_{\alpha\beta}\gamma_{m\ \gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta \tag{2.14}$$

is the super-Poincaré covariant Green-Schwarz constraint[7]. Note that $Q^2 = 0$ since d_α satisfies the OPE's

$$d_\alpha(y)d_\beta(z) \to -(y-z)^{-1}\gamma^m_{\alpha\beta}\Pi_m(z)\,, \qquad d_\alpha(y)\Pi^m(z) \to (y-z)^{-1}\gamma^m_{\alpha\beta}\partial\theta^\beta(z)\,, \tag{2.15}$$

where $\Pi^m = \partial x^m + \frac{1}{2}\theta\gamma^m\partial\theta$ is the supersymmetric momentum. Furthermore, Q is spacetime supersymmetric since d_α anticommutes with the spacetime supersymmetry generators[7]

$$q_\alpha = \oint \left(p_\alpha + \frac{1}{2}\gamma^m_{\alpha\beta}\theta^\beta\partial x_m + \frac{1}{24}\gamma^m_{\alpha\beta}\gamma_{m\ \gamma\delta}\theta^\beta\theta^\gamma\partial\theta^\delta\right). \tag{2.16}$$

For Q to be hermitean, λ^α must be defined to be a hermitean operator. Although the pure spinor condition of (2.1) has no real non-vanishing solutions, this does not cause any inconsistency. Since the Hilbert space inner product does not have a positive definite norm, there is no reason why $(\lambda^\alpha)^\dagger\lambda^\alpha$ must be an operator with positive eigenvalues [11]. Note, however, that when λ^α is hermitean, the (s, t, u_{ab}, v^{ab}) variables have strange hermiticity properties. For example, using the notation of footnote 5, $(\lambda^{+++++})^\dagger = \lambda^{+----}$ implies that $(e^s)^\dagger = -\frac{1}{8}e^{-s}\epsilon^{abcd5}u_{ab}u_{cd}$.

The physical vertex operator for the massless super-Maxwell multiplet is $U = \lambda^\alpha A_\alpha(x,\theta)$ where $A_\alpha(x,\theta)$ is the spinor gauge potential of super-Maxwell theory. $QU = 0$ and $\delta U = Q\Lambda$ implies that $D_\alpha(\gamma_{mnpqr})^{\alpha\beta}A_\beta = 0$ and $\delta A_\alpha = D_\alpha\Lambda$ where $D_\alpha = \frac{\partial}{\partial\theta^\alpha} + \frac{1}{2}\theta^\beta\gamma^m_{\alpha\beta}\partial_m$ and $mnpqr$ is an arbitrary five-form direction [8]. These are the superspace equations of motion and gauge invariances of super-Maxwell theory.

Using the λ^α parameterization defined in (2.4), pure spinor vertex operators can be described in terms of unconstrained variables. However, not every function of the unconstrained variables is an allowed pure spinor vertex operator. The requirement of super-Poincaré covariance implies that the function must transform as a finite dimensional representation under commutation with the Lorentz generators of (2.7), which implies that the (s, t, u_{ab}, v^{ab}) variables can only appear in the Lorentz covariant combinations $(\lambda^\alpha, N_{mn}, \partial h)$ of (2.4), (2.7), and (2.12).

2.3 Tree amplitudes

To compute N-point tree amplitudes, one needs 3 dimension-zero vertex operators U and $N-3$ dimension-one vertex operators V which will be integrated over the real line. For a given physical state described by the dimension-zero operator U, the dimension-one operator V can be defined by requiring that $QV = \partial U$. For the super-Maxwell vertex operator, one can check that

$$V = \partial\theta^\alpha A_\alpha(x,\theta) + \Pi^m B_m(x,\theta) + d_\alpha W^\alpha(x,\theta) + \frac{1}{2}N^{mn}F_{mn}(x,\theta) \tag{2.17}$$

where $B_m = \frac{1}{8}D_\alpha\gamma_m^{\alpha\beta}A_\beta$, $W^\beta = \frac{1}{10}\gamma_m^{\alpha\beta}(D_\alpha B^m - \partial^m A_\alpha)$, and $F_{mn} = \frac{1}{8}D_\alpha(\gamma_{mn})^\alpha{}_\beta W^\beta = \partial_{[m}B_{n]}$.

Tree amplitudes are then defined by the correlation function

$$A = \int dz_4\cdots dz_N \left\langle\, U_1(z_1)\, U_2(z_2)\, U_3(z_3) \int dz_4 V_4(z_4)\cdots\int dz_N V_N(z_N)\right\rangle. \tag{2.18}$$

As shown in [3], this definition is independent of which three external states are represented by unintegrated vertex operators. Since the action of (2.2) is quadratic, the free field OPE's can be used to perform the integration over the non-zero modes of the worldsheet fields. The resulting expression,

$$\begin{aligned} A &= \int dz_4\cdots dz_N \langle\, f(k_r, \eta_r, z_r, \lambda, \theta)\,\rangle \\ &= \int dz_4\cdots dz_N \langle\, \lambda^\alpha\lambda^\beta\lambda^\gamma f_{\alpha\beta\gamma}(k_r, \eta_r, z_r, \theta)\,\rangle\,, \end{aligned} \tag{2.19}$$

only depends on the external momenta k_r and polarizations η_r, and on the zero modes of the λ^α and θ^α fields. Furthermore, the expression is cubic in λ^α since U carries ghost-number $+1$ and V carries ghost-number 0.

JHEP08(2001)026

The (λ, θ) zero-mode integration will be defined by

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}(k_r, \eta_r, z_r, \theta)\rangle = \left(\frac{\partial}{\partial\theta}\gamma_{mnp}\frac{\partial}{\partial\theta}\right)\left(\frac{\partial}{\partial\theta}\gamma^m\right)^\alpha \left(\frac{\partial}{\partial\theta}\gamma^n\right)^\beta \times$$
$$\times \left(\frac{\partial}{\partial\theta}\gamma^p\right)^\gamma f_{\alpha\beta\gamma}(k_r, \eta_r, z_r, \theta)|_{\theta=0}\,. \tag{2.20}$$

In other words, only the term proportional to $(\theta\gamma^{mnp}\theta)(\gamma_m\theta)_\alpha(\gamma_n\theta)_\beta(\gamma_p\theta)_\gamma$ in $f_{\alpha\beta\gamma}$ contributes to the scattering amplitude. Since $(\theta\gamma^{mnp}\theta)(\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)$ does not equal $Q\Lambda$ for any Lorentz covariant Λ, the zero mode prescription of (2.20) implies that

$$\langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma} + Q\Lambda\rangle = \langle \lambda^\alpha \lambda^\beta \lambda^\gamma f_{\alpha\beta\gamma}\rangle\,, \tag{2.21}$$

so the amplitude is gauge invariant. This tree amplitude prescription was shown in [1] to be spacetime supersymmetric and was shown by explicit computation in [3] to coincide with the RNS tree amplitude prescription for massless external states with up to four fermions, In subsection 5.4 of this paper, the map from pure spinor vertex operators to RNS vertex operators will be used to argue that this tree amplitude prescription agrees with the RNS prescription for arbitrary external states.

3. RNS picture-changing as a gauge transformation

In this section, we shall discuss the relevant features of the RNS formalism using a modified definition of physical states where picture-changing is a gauge transformation.

3.1 Picture-changing

Physical vertex operators in the RNS formalism can be defined as ghost-number +1 operators[7] U which satisfy [13]

$$\eta_0 U = Q_{RNS} U = 0\,, \qquad \delta U = Q_{RNS}\eta_0\Lambda\,, \tag{3.1}$$

where

$$Q_{RNS} = \oint\Big[c\Big(-\frac{1}{2}\partial x^m\partial x_m - \frac{1}{2}\psi^m\partial\psi_m - \eta\partial\xi - \frac{1}{2}\partial\phi\partial\phi - \partial^2\phi - b\partial c\Big) +$$
$$+ \eta e^\phi\psi^m\partial x_m - \eta\partial\eta e^{2\phi} b\Big] \tag{3.2}$$

is the RNS BRST operator and η_0 is the zero mode coming from fermionizing the (β, γ) super-reparameterization ghosts as $\beta = \partial\xi e^{-\phi}$ and $\gamma = \eta e^\phi$ [4]. Note that $\eta_0 U = 0$ implies that U is in the "small" Hilbert space and that any gauge parameter in the "small" Hilbert space, Ω, can be written as $\Omega = \eta_0\Lambda$ for some Λ.

[7] As in [12], we shall define the ghost-number operator as $\oint(cb + \eta\xi)$ so that η carries ghost-number +1. This definition agrees with the ghost-number operator $\oint(cb - \partial\phi)$ of [4] at zero picture, but has the advantage of commuting with picture-changing and spacetime-supersymmetry.

It might seem surprising that Q_{RNS} and η_0 appear symmetrically in the definition of (3.1). However, note that in the "large" Hilbert space, both Q_{RNS} and η_0 have trivial cohomologies. The cohomology of Q_{RNS} is trivial in the "large" Hilbert space since any state U satisfying $Q_{RNS}U = 0$ can be written as $U = Q_{RNS}(c\xi\partial\xi e^{-2\phi}U)$. And the cohomology of η_0 is trivial in the "large" Hilbert space since any state U satisfying $\eta_0 U = 0$ can be written as $U = \eta_0(\xi U)$.

Using the picture-raising operation,

$$U_{P+1} = \{Q_{RNS}, \xi U_P\} \tag{3.3}$$

or the picture-lowering operation,

$$U_{P-1} = \{\eta_0, c\xi\partial\xi e^{-2\phi}U_P\}\,, \tag{3.4}$$

any physical state represented by the vertex operator U_P of picture P can also be represented by a vertex operator of arbitrarily higher or lower picture. As will now be shown, this redundancy in describing physical states can be removed by defining physical vertex operators as ghost-number $+1$ operators in the cohomology of $Q' = Q_{RNS} + \eta_0$.

3.2 Cohomology of Q'

To show that the cohomology of $Q' = Q_{RNS} + \eta_0$ correctly reproduces the physical spectrum, consider a vertex operator U annihilated by Q' of the form $U = \sum_{P=P_-}^{P_+} U_P$ where U_P carries picture P and $P_\pm$ are the highest/lowest picture.[8] Since Q_{RNS} carries zero picture and η_0 carries -1 picture, $Q'U = 0$ implies that $\eta_0 U_{P_-} = 0$, which implies that $U_{P_-} = \eta_0 \Omega_{P_-+1}$ for some Ω_{P_-+1} of picture $P_- + 1$. Then after the gauge transformation $\delta U = -Q'\Omega_{P_-+1}$, $U = \sum_{P=P_-+1}^{P_+} U_P$ where U_{P_-+1} is now $U_{P_-+1} - Q_{RNS}\Omega_{P_-+1}$. This procedure can be continued until $U = U_{P_+}$ where $Q'U_{P_+} = 0$. But this implies that $Q_{RNS}U_{P_+} = \eta_0 U_{P_+} = 0$, so U_{P_+} is a physical state using the definition of (3.1).

To show that this physical state is not pure gauge, suppose that $U = U_{P_+} = Q'\Lambda$ for some Λ. Then $\Lambda = \Lambda_{P_+} + \Lambda_{P_++1}$ where

$$U = Q_{RNS}\Lambda_{P_+} + \eta_0\Lambda_{P_++1}\,, \tag{3.5}$$

$$Q_{RNS}\Lambda_{P_++1} = \eta_0\Lambda_{P_+} = 0\,. \tag{3.6}$$

The Q_{RNS} and η_0 cohomologies are trivial in the "large" Hilbert space so (3.6) implies that $\Lambda_{P_++1} = Q_{RNS}(c\xi\partial\xi e^{-2\phi}\Lambda_{P_++1})$ and $\Lambda_{P_+} = \eta_0(\xi\Lambda_{P_+})$. Plugging into (3.5), one finds

$$U = Q_{RNS}\eta_0(-c\xi\partial\xi e^{-2\phi}\Lambda_{P_++1} + \xi\Lambda_{P_+})\,,$$

which implies that U would have been pure gauge using the definition of (3.1).

[8] Only vertex operators involving a finite number of picture will be allowed in the Hilbert space. If vertex operators involving arbitrarily high (or low) picture were allowed in the Hilbert space, the cohomology of Q' would be trivial.

So any state in the cohomology of $Q' = Q_{RNS} + \eta_0$ determines a physical vertex operator using the definition of (3.1). Also, any two physical vertex operators which are related by the picture changing operations of (3.3) or (3.4) are described by the same state in the cohomology of Q'. For example, the vertex operator U and the picture-raised operator $\{Q_{RNS}, \xi U\}$ are related by the gauge transformation $\delta U = Q'(\xi U)$. Similarly, the vertex operator U and the picture-lowered operator $\{\eta_0, c\xi\partial\xi e^{-2\phi}U\}$ are related by the gauge transformation $\delta U = Q'(c\xi\partial\xi e^{-2\phi}U)$.

4. Relating the RNS and pure spinor BRST operators

In this section, the operator $Q' = Q_{RNS} + \oint \eta$ of section 3 will be related to the operator $Q = \oint \lambda^\alpha d_\alpha$ of section 2. This will be done by first finding a field redefinition which maps Q' to a spacetime supersymmetric operator $Q_{\mathrm{U}(5)}$, and then constructing a similarity transformation R which satisfies $e^{-R}Q_{\mathrm{U}(5)}e^R = Q$.

Because $\{q_\alpha^{-1/2}, q_\beta^{-1/2}\} = \gamma^m_{\alpha\beta}\oint e^{-\phi}\psi_m$ in the RNS formalism where $q_\alpha^{-1/2} = \oint e^{-1/2\phi}\Sigma_\alpha$ is the spacetime supersymmetry generator in the $-1/2$ picture and Σ_α is the spin field, the spacetime supersymmetry algebra only closes up to picture changing in the RNS formalism. Nevertheless, since $\{q_\alpha^{-1/2}, q_\beta^{+1/2}\} = \gamma^m_{\alpha\beta}\oint \partial x_m$ where $q_\beta^{+1/2} = \oint(b\eta e^{\frac{3}{2}\phi}\Sigma_\beta + e^{\frac{1}{2}\phi}\gamma^m_{\alpha\beta}\Sigma^\alpha\partial x_m)$ is the spacetime supersymmetry generator in the $+1/2$ picture, one can make a subset of the algebra close by choosing some of the supersymmetry generators in the $-1/2$ picture and others in the $+1/2$ picture. However, this choice necessarily breaks manifest Lorentz invariance. As shown in [6] and will be reviewed in the following subsection, U(5) is the maximum subgroup of the (Wick-rotated) SO(10) Lorentz group which can be manifestly preserved, which is the subgroup of SO(10) that leaves a pure spinor invariant. Other choices for the subgroup are useful for describing Calabi-Yau compactifications of the superstring [14][13].

4.1 Review of U(5) formalism

Under U(5), the SO(10) spinor q_α splits into (q_+, q^{ab}, q_a) which transform as $(1_{-5/2}, 10_{-1/2}, \overline{5}_{3/2})$ representations of $SU(5)_{\mathrm{U}(1)}$. If q_+ is chosen in the $+1/2$ picture and q_a is chosen in the $-1/2$ picture, these six generators preserve the supersymmetry algebra since $\{q_a, q_b\} = \{q_+, q_+\} = 0$.

With this choice of picture, it is natural to define $\theta^a = e^{\frac{1}{2}\phi}\Sigma^a$ and $\theta^+ = c\xi e^{-\frac{3}{2}\phi}\Sigma^+$ so that $\{q_a, \theta^b\} = \delta_a^b$ and $\{q_+, \theta^+\} = 1$. One then defines conjugate momenta to θ^a and θ^+ by $p_a = e^{-\frac{1}{2}\phi}\Sigma_a$ and $p_+ = b\eta e^{\frac{3}{2}\phi}\Sigma_+$. Since the RNS variables include twelve fermions (ψ^m, b, c) and two chiral bosons (β, γ), there are still two independent chiral bosons which will be defined as[9]

$$\partial s = -bc - \frac{3}{2}\partial\phi - \frac{1}{2}\sum_{a=1}^{5}\psi^{2a-2}\psi^{2a-1}\,, \qquad \partial t = -\xi\eta + \frac{3}{2}\partial\phi + \frac{1}{2}\sum_{a=1}^{5}\psi^{2a-2}\psi^{2a-1}\,.$$

From the RNS OPE's, one finds that the only singular OPE's of $(x^m, \theta^+, \theta^a, p_+, p_a, s, t)$ are

$$x^m(y)x^n(z) \to -\eta^{mn}\log|y-z|^2\,,$$
$$\theta^+(y)p_+(z) \to (y-z)^{-1}\,, \quad \theta^a(y)p_b(z) \to \delta^a_b(y-z)^{-1}\,, \quad s(y)t(z) \to \log(y-z)\,.$$

In terms of these GS-like variables, the RNS stress tensor is

$$T_{RNS} = -\frac{1}{2}\partial x^m\partial x_m - p_+\partial\theta^+ - p_a\partial\theta^a + \partial s\partial t + \partial^2 s\,, \tag{4.1}$$

the RNS BRST operator is

$$Q_{RNS} = \oint\left(e^t\left[p_a\partial x^a - \theta^+\partial x_a\partial x^a + p_a\theta^+\partial\theta^a + \partial\theta^+\left(\partial s + \frac{1}{2}\partial t\right)\right] - \right.$$
$$\left. - \frac{1}{120}e^{2t-s}\epsilon^{abcde}(p_a p_b p_c p_d p_e - 5\theta^+ p_a p_b p_c p_d \partial x_e)\right)\,, \tag{4.2}$$

the RNS b and η ghosts are

$$b_{RNS} = e^{-t}p_+\,, \qquad \eta = e^s p_+\,, \tag{4.3}$$

and the RNS ghost-number current is

$$j_{RNS} = cb + \eta\xi = \partial s + \partial t\,, \tag{4.4}$$

where the SO(10) vector x^m has been split into its U(5) components as x^a and x_a which transform as 5_{+1} and $\overline{5}_{-1}$ representations of $SU(5)_{U(1)}$ and satisfy the OPE $x^a(y)x_b(z) \to -\delta^b_a\log|y-z|^2$.

To relate $Q' = Q_{RNS} + \oint\eta$ with $Q = \oint\lambda^\alpha d_\alpha$, it will be convenient to first perform a unitary transformation on the GS-like variables (p_+, p_a, x^a) such that

$$p_+^{new} = p_+^{old} + \frac{1}{2}\theta^a\partial x_a\,, \qquad p_a^{new} = p_a^{old} - \frac{1}{2}\theta^+\partial x_a\,, \qquad x^a_{new} = x^a_{old} - \frac{1}{2}\theta^a\theta^+\,. \tag{4.5}$$

In terms of the "new" GS-like variables, one can check that

$$Q' = Q_{RNS} + \oint\eta$$
$$= \oint\left(e^s\widehat{d}_+ + e^t\left[\widehat{d}_a\widehat{\Pi}^a + \partial\theta^+\left(\partial s - \frac{3}{4}\partial t\right)\right] - \frac{1}{120}e^{2t-s}\epsilon^{abcde}\widehat{d}_a\widehat{d}_b\widehat{d}_c\widehat{d}_d\widehat{d}_e\right)\,, \tag{4.6}$$

[9]These two chiral bosons are related to ρ and σ of [6] by $s = \frac{1}{2}(-\rho + i\sigma)$ and $t = \frac{1}{2}(\rho + i\sigma)$.

JHEP08(2001)026

where

$$\widehat{d}_+ = p_+ - \frac{1}{2}\theta^a \partial x_a\,, \qquad \widehat{d}_a = p_a - \frac{1}{2}\theta^+ \partial x_a\,, \qquad \widehat{\Pi}^a = \partial x^a + \frac{1}{2}(\theta^a \partial\theta^+ + \theta^+ \partial\theta^a)\,, \quad (4.7)$$

are defined like d_α and Π_m in (2.14) and (2.15) but with θ_{ab} set to zero. The operator of (4.6) is manifestly invariant under the six supersymmetry transformations generated by $\widehat{q}_a = \oint (p_a + \frac{1}{2}\theta^+ \partial x_a)$ and $\widehat{q}_+ = \oint (p_+ + \frac{1}{2}\theta^a \partial x_a)$.

So using the arguments of section 3, physical RNS states can be described in U(5) language as states in the cohomology of the operator (4.6). After adding a topological sector to the U(5) formalism, it will be shown in the next subsection how to relate Q' with the pure spinor BRST operator $Q = \oint \lambda^\alpha d_\alpha$.

4.2 Supersymmetric U(5) formalism

The cohomology of Q' of (4.6) defines physical states in a manner which manifestly preserves six spacetime supersymmetries. As will now be shown, all sixteen supersymmetries can be made manifest if one adds ten new fermionic variables and their conjugates, (θ_{ab}, p^{ab}), as well as ten new bosonic variables and their conjugates, (u_{ab}, v^{ab}), to the U(5) variables of the previous subsection. These new variables are not related to RNS worldsheet variables so the BRST operator must be modified such that the new variables do not affect the physical states. Note that a similar trick was used in the six-dimensional version of the hybrid formalism where four θ variables and their conjugates were added in order to make all eight spacetime supersymmetries manifest [15].

The first step is to change the RNS action to

$$S_{\mathrm{U}(5)} = S_{RNS} + \frac{1}{2}\int d^2 z\, \left(p^{ab}\overline{\partial}\theta_{ab} - v^{ab}\overline{\partial} u_{ab}\right)$$

so that the new variables satisfy the OPE's

$$p^{ab}(y)\theta_{cd}(z) \to \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}\,, \qquad v^{ab}(y)u_{cd}(z) \to \delta_c^{[a}\delta_d^{b]}(y-z)^{-1}\,.$$

One now modifies the BRST operator to

$$Q_{\mathrm{U}(5)} = \lambda^\alpha d_\alpha + e^t\left(d_a \Pi^a + \partial\theta^+\left(\partial s - \frac{3}{4}\partial t\right)\right) + \frac{1}{12}e^{t-s}\epsilon^{abcde}u_{ab}d_c d_d d_e - e^{2t-s}(d)^5 \qquad (4.8)$$

where λ^α is defined in (2.4), d_α and Π^m are defined in (2.14) and (2.15), and $(d)^5 = \frac{1}{120}\epsilon^{abcde}d_a d_b d_c d_d d_e$. Using the OPE's of (2.15), one can check that $Q_{\mathrm{U}(5)}$ of (4.8) is nilpotent.[10] It will also be convenient to modify the b ghost and ghost-number

[10] A useful trick for computations is to perform a unitary transformation analogous to (4.5) such that $d_a = p_a$. This unitary transformation changes only the term $e^t\partial\theta^+(\partial s - \frac{3}{4}\partial t)$ to $e^t\partial\theta^+(\partial s - 2\partial t)$ in $Q_{\mathrm{U}(5)}$ and simplifies OPE's involving d_a. For example, after the transformation, $d_+(y)d_a d_b(z) \to (y-z)^{-2}\partial\theta_{ab}(y) - (y-z)^{-1}\Pi_{[a}(y)d_{b]}(z) + d_+ d_a d_b(z)$.

current of (4.3) and (4.4) to

$$b_{\mathrm{U}(5)} = e^{-t}d_+ + \frac{1}{2}v^{ab}\partial\theta_{ab}\,, \qquad j_{\mathrm{U}(5)} = \partial s + \partial t + \frac{1}{2}u_{ab}v^{ab}\,, \tag{4.9}$$

so the stress tensor of (4.1) is modified to

$$\begin{aligned} T_{\mathrm{U}(5)} &= \{Q_{\mathrm{U}(5)}, b_{\mathrm{U}(5)}\} = T_{RNS} - \frac{1}{2}p^{ab}\partial\theta_{ab} + \frac{1}{2}v^{ab}\partial u_{ab} \\ &= -\frac{1}{2}\partial x^m\partial x_m - p_\alpha\partial\theta^\alpha + \frac{1}{2}v^{ab}\partial u_{ab} + \partial t\partial s + \partial^2 s\,. \end{aligned} \tag{4.10}$$

It will now be shown that $Q_{\mathrm{U}(5)}$ in the enlarged Hilbert space has the same cohomology as Q' of (4.6) in the old Hilbert space without the new variables. To relate the $Q_{\mathrm{U}(5)}$ and Q' cohomologies, first write

$$Q_{\mathrm{U}(5)} = \frac{1}{2}\oint u_{ab}p^{ab} + Q' + f(u_{ab},\theta_{ab}) \tag{4.11}$$

where $f(u_{ab},\theta_{ab})$ includes all terms in $Q_{\mathrm{U}(5)}$ except $\frac{1}{2}\oint u_{ab}p^{ab}$ which involve u_{ab} or θ_{ab}. If (p^{ab},θ_{ab}) are assigned "charge" $(-2,+2)$, (v^{ab},u_{ab}) are assigned "charge" $(-1,+1)$, and all other variables are assigned zero "charge", then the terms of (4.11) are written in order of increasing "charge", i.e. $Q_{\mathrm{U}(5)} = Q_{(-1)} + Q_{(0)} + Q_{(1)} + \cdots$ where $Q_{(n)}$ carries "charge" n.[11]

Now consider a vertex operator $U_{\mathrm{U}(5)} = \sum_{n=C}^{\infty} U_{(n)}$ where n denotes the "charge" of the term $U_{(n)}$ and $U_{(C)}$ is the term of lowest "charge". Note that C is bounded from below since only variables of positive conformal weight carry negative "charge". Suppose that $U_{\mathrm{U}(5)}$ is in the cohomology of $Q_{\mathrm{U}(5)}$, i.e. $Q_{\mathrm{U}(5)}U_{\mathrm{U}(5)} = 0$ and $\delta U_{\mathrm{U}(5)} = Q_{\mathrm{U}(5)}\Lambda$ for $\Lambda = \sum_n \Lambda_{(n)}$. Then $Q_{(-1)}U_{(C)} = 0$ and $\delta U_{(C)} = Q_{(-1)}\Lambda_{(C+1)}$ where $Q_{(-1)} = \frac{1}{2}\oint u_{ab}p^{ab}$. So using the quartet mechanism, $U_{(C)}$ can be gauge-fixed to be independent of the new variables $(u_{ab}, v^{ab}, \theta_{ab}, p^{ab})$. But since "charge" is carried only by the new variables, one can choose a gauge such that $U_{(C)} = 0$ if $C < 0$. Similarly, one can choose a gauge such that $U_{(n)} = 0$ for all $n < 0$ and such that $U_{(0)}$ is independent of the new variables.

In this gauge, $Q_{(0)}U_{(0)} = 0$ and $\delta U_{(0)} = Q_{(0)}\Lambda_{(0)}$ where $U_{(0)}$ and $\Lambda_{(0)}$ are independent of the new variables and $Q_{(0)} = Q'$. In other words, $U_{(0)}$ describes states in the cohomology of Q' in the Hilbert space without the new variables. Finally, it will be shown by induction that all terms $U_{(n)}$ for $n > 0$ are determined by $U_{(0)}$ up to a gauge transformation.

Suppose that $U_{(n)}$ is known for $0 \le n \le M$. Then $Q_{(-1)}Q_{\mathrm{U}(5)}U_{\mathrm{U}(5)} = 0$ implies that $Q_{(-1)}(\sum_{n=0}^{M} Q_{(n)}U_{(M-n)}) = 0$. But since $Q_{(-1)} = \frac{1}{2}\oint u_{ab}p^{ab}$ has trivial cohomology at non-zero "charge", there exists an operator $U_{(M+1)}$ satisfying $Q_{(-1)}U_{(M+1)} = -\sum_{n=0}^{M} Q_{(n)}U_{(M-n)}$ which is uniquely determined up to the gauge

[11] I would like to thank Edward Witten for suggesting this method for analyzing the cohomology.

transformation $\delta U_{(M+1)} = Q_{(-1)}\Lambda_{(M+2)}$. Similarly, all terms $U_{(n)}$ for $n > 0$ are determined by $U_{(0)}$ up to a gauge transformation. This completes the proof that the cohomology of $Q_{\mathrm{U}(5)}$ of (4.8) in the enlarged Hilbert space is equivalent to the cohomology of Q' of (4.6) in the Hilbert space without the new variables.

4.3 Similarity transformation

The next step in relating Q' and $Q = \oint \lambda^\alpha d_\alpha$ is to find a similarity transformation R such that $e^{-R} Q_{\mathrm{U}(5)} e^R = \oint \lambda^\alpha d_\alpha$ where λ^α is defined in (2.4). This similarity transformation is

$$R = \oint \left(e^{t+s} g_a \Pi^a - \frac{1}{4} \epsilon^{abcde} e^t g_a u_{bc} d_d d_e \right) \tag{4.12}$$

where g_a is defined to be any function of u_{ab} which satisfies

$$\frac{1}{8} e^{abcde} g_a u_{bc} u_{de} = 1\,. \tag{4.13}$$

To preserve (4.13), g_a is defined to satisfy the OPE

$$g_a(y) v^{bc}(z) \to \frac{1}{2}(y-z)^{-1} \epsilon^{bcdef} g_a g_d u_{ef}(z)\,. \tag{4.14}$$

For example, if $\epsilon^{abcd5} u_{ab} u_{cd}$ is non-zero, one can choose $g_a = 8\ \delta_a^5 (\epsilon^{bcde5} u_{bc} u_{de})^{-1}$, which can be checked to satisfy (4.14). Note that (4.13) has no solutions if all five components of $\epsilon^{abcde} u_{bc} u_{de}$ are zero, i.e. if $\lambda^a = 0$. As will be discussed in subsection 5.2, this creates subtleties when using the similarity transformation of R to relate U(5) and pure spinor vertex operators.[12] To check that $e^{-R} Q_{\mathrm{U}(5)} e^R = \oint \lambda^\alpha d_\alpha$, note that

$$[R, \lambda^\alpha d_\alpha] = e^t \left(d_a \Pi^a + \partial\theta^+ \left(\partial s - \frac{3}{4} \partial t \right) \right) + \frac{1}{12} e^{t-s} \epsilon^{abcde} u_{ab} d_c d_d d_e\,,$$
$$[R, [R, \lambda^\alpha d_\alpha]\,] = -2 e^{2t-s} (d)^5\,, \qquad \text{and} \qquad [R, [R, [R, \lambda^\alpha d_\alpha]\,]\,] = 0\,. \tag{4.15}$$

So the RNS operator $Q' = Q_{RNS} + \oint \eta$ has been mapped to the pure spinor operator $Q = \oint \lambda^\alpha d_\alpha$ using a field redefinition and similarity transformation. In the next section, this map will be used to relate vertex operators and tree amplitudes in the two formalisms.

5. Relating the RNS and pure spinor vertex operators

5.1 Relating RNS and U(5) vertex operators

As discussed in subsections 4.1 and 4.2, the RNS worldsheet variables can be related by a field redefinition to the GS-like variables $(x^m, \theta^+, \theta^a, p_+, p_a, s, t)$, which can then

[12] These subtleties are reminiscent of the subtleties with the λ^α parameterization of (2.4) when $\lambda^+ = 0$. Perhaps these subtleties can be avoided by using transition functions to relate different patches of λ^α space in a manner analogous to the conventional treatment of Penrose's twistor space [16].

JHEP08(2001)026

be covariantized by adding the "topological" variables $(\theta_{ab}, p^{ab}, u_{ab}, v^{ab})$. Physical states in this U(5) Hilbert space are defined as states in the cohomology of $Q_{\mathrm{U}(5)}$ of (4.8).

As was shown in subsection 4.2, any physical RNS vertex operator U_{RNS} can be mapped to a vertex operator $U_{\mathrm{U}(5)}$ in the cohomology of $Q_{\mathrm{U}(5)}$ by using the field redefinition to write U_{RNS} in terms of U(5) variables and defining $U_{\mathrm{U}(5)} = U_{RNS} + \cdots$ where $\cdots$ involves operators of positive "charge" which are determined by U_{RNS} up to a gauge transformation. Similarly, any physical vertex operator $U_{\mathrm{U}(5)}$ can be mapped to a vertex operator U_{RNS} in the cohomology of $Q' = Q_{RNS} + \oint \eta$ by first choosing a gauge in which $U_{\mathrm{U}(5)}$ has no operators with negative "charge". One can then define U_{RNS} as the operator in $U_{\mathrm{U}(5)}$ with zero "charge" and use the field redefinition to write this operator in terms of RNS variables.

In order to complete the map between physical RNS and physical pure spinor vertex operators, one therefore needs to find a map between physical U(5) vertex operators and pure spinor vertex operators in the cohomology of $Q = \oint \lambda^\alpha d_\alpha$.

5.2 Relating U(5) and pure spinor vertex operators

Although $Q = e^{-R} Q_{\mathrm{U}(5)} e^R$ where R is defined in (4.12), this similarity transformation cannot be directly used to relate physical pure spinor and U(5) vertex operators by $U = e^{-R} U_{\mathrm{U}(5)} e^R$ and $U_{\mathrm{U}(5)} = e^R U e^{-R}$ since R does not preserve the relevant Hilbert spaces. Note that both the U(5) and pure spinor Hilbert spaces consist of functions constructed from the variables $(x^m, \theta^\alpha, p_\alpha, s, t, u_{ab}, v^{ab})$. However, vertex operators in the pure spinor Hilbert space are required to be Lorentz-covariant functions of (s, t, u_{ab}, v^{ab}), i.e. functions which are polynomials in λ^α, N_{mn} and ∂h of (2.4), (2.7), and (2.12). But $e^{-R} U e^R$ does not necessarily have this property since $Q_{\mathrm{U}(5)}$ and R are not Lorentz invariant. Also, vertex operators in the U(5) Hilbert space must be independent of g_a so that they are well-defined when $\epsilon^{abcde} u_{bc} u_{de} = 0$. So because of explicit g_a dependence in R, $e^R U e^{-R}$ is not necessarily an allowable U(5) vertex operator.

Although $Q_{\mathrm{U}(5)}$ and R are not super-Poincaré invariant, they are manifestly invariant under all sixteen spacetime supersymmetry transformations and under a U(5) subgroup of the (Wick-rotated) SO(10) Lorentz transformations. It will be convenient to define operators which transform covariantly under this subgroup of super-Poincaré transformations as "almost super-Poincaré covariant" operators, or ASPC operators for short.

It will now be conjectured that for each physical pure spinor vertex operator U, one can define a physical U(5) vertex operator $U_{\mathrm{U}(5)}$ by

$$U_{\mathrm{U}(5)} = e^R U e^{-R} + Q_{\mathrm{U}(5)} \Omega \tag{5.1}$$

where Ω is some ASPC operator which is allowed to depend on g_a. Furthermore, it will be conjectured that different ASPC choices for Ω only change $U_{\mathrm{U}(5)}$ by a U(5) gauge transformation, i.e. by $Q_{\mathrm{U}(5)}\Lambda_{\mathrm{U}(5)}$ where $\Lambda_{\mathrm{U}(5)}$ is independent of g_a.

It is crucial that Ω is restricted to be an ASPC operator since otherwise, the map could relate physical vertex operators with BRST-trivial vertex operators. For example, $Q_{\mathrm{U}(5)}U_{\mathrm{U}(5)} = 0$ implies that $U_{\mathrm{U}(5)} = Q_{\mathrm{U}(5)}(-e^s\theta^a g_a U_{\mathrm{U}(5)})$ and $QU = 0$ implies that $U = Q(e^{-s}\theta^+ U)$. But $e^s\theta^a g_a U_{\mathrm{U}(5)}$ and $e^{-s}\theta^+ U$ are not ASPC operators since they contain explicit dependence on θ^α zero modes. Note that to transform covariantly under spacetime supersymmetry, the ASPC operator Ω must be constructed from products of spacetime-supersymmetric operators and covariantly transforming spacetime superfields which appear in U.

Since R is proportional to e^t and and pure spinor operators U have no e^t dependence, one can always choose Ω of (5.1) such that it has only positive powers of e^t. In this case,

$$U_{\mathrm{U}(5)} = U + U_{ASPC}\,, \tag{5.2}$$

where U is a physical pure spinor vertex operator and U_{ASPC} is an ASPC operator which is independent of g_a and contains only positive powers of e^t. Any physical U(5) vertex operator of the form of (5.2) will be called a "special" U(5) vertex operator.

Since the RNS and pure spinor cohomologies were shown to be equivalent in [2], the conjecture of (5.1) implies that any state in the cohomology of $Q_{\mathrm{U}(5)}$ can be represented by a "special" U(5) vertex operator.[13] Furthermore, the conjecture implies that any two "special" vertex operators with the same pure spinor vertex operator as their e^t-independent component are related by a U(5) gauge transformation. So the conjecture of (5.1) not only implies a map from physical pure spinor vertex operators to physical U(5) vertex operators, but also implies (with the exception of the zero momentum state mentioned in footnote 13) a map using (5.2) from physical U(5) vertex operators to physical pure spinor vertex operators.

Evidence for the conjecture of (5.1) will now be obtained by explicitly constructing "special" U(5) vertex operators for the physical massless states and by using the property that "special" massive vertex operators can be obtained from the OPE's of "special" massless vertex operators.

5.3 "Special" massless vertex operators

In the pure spinor formalism where $Q = \oint \lambda^\alpha d_\alpha$, the physical unintegrated massless vertex operator of ghost-number +1 is $U = \lambda^\alpha A_\alpha(x,\theta)$ where $D_{(\alpha}A_{\beta)} = \gamma^m_{\alpha\beta}B_m$ and $D_\alpha B^m - \partial^m A_\alpha = \gamma^m_{\alpha\beta}W^\beta$. To compute $U_{\mathrm{U}(5)}$, it will be convenient to first use the

[13] More precisely, the cohomologies at non-zero P_+ were shown to be equivalent. As will be discussed in section 6, the RNS cohomology has an extra zero-momentum state at ghost-number -1 which cannot be represented by a "special" U(5) vertex operator.

gauge invariance $\delta A_\alpha = D_\alpha\Lambda$ to gauge-fix $A_a = 0$. In this gauge,

$$
\begin{aligned}
e^R(\lambda^\alpha A_\alpha)e^{-R} &= e^R\left(e^s A_+ + \frac{1}{2}u_{ab}A^{ab}\right)e^{-R}\\
&= e^s A_+ + \frac{1}{2}u_{ab}A^{ab} - \frac{1}{2}\epsilon^{abcde}e^t g_a u_{bc} d_d u_{ef} B^f -\\
&\quad - \frac{1}{4}\epsilon^{abcde}\partial(e^t g_a u_{bc})u_{de}W^+ + e^{s+t}\left(\frac{1}{4}\epsilon^{abcde}g_a u_{bc}W_{de} + \frac{1}{2}g_a u_{bc}\partial^a A^{bc}\right) +\\
&\quad + \frac{1}{4}e^{2t}\epsilon^{abcde}g_a u_{bc} d_d d_e W^+ + e^{2t+s}g_a\partial^a W^+\\
&= e^s A_+ + \frac{1}{2}u_{ab}A^{ab} + e^t(d_a B^a + \partial(s-2t)W^+) + Q_{\mathrm{U}(5)}(e^{t+s}g_a B^a)\,,
\end{aligned}
\tag{5.3}
$$

where A_α, B_m and W^α have been written in terms of their U(5)-irreducible components.

So for massless states, the conjecture of (5.1) has been confirmed where in the gauge $A_a = 0$, $U_{\mathrm{U}(5)} = \lambda^\alpha A_\alpha + e^t(d_a B^a + \partial(s-2t)W^+)$ and $\Omega = e^{t+s}g_a B^a$. To construct $U_{\mathrm{U}(5)}$ in other gauges, use the fact that $\delta U_{\mathrm{U}(5)} = Q_{\mathrm{U}(5)}\Lambda$ where $\delta A_\alpha = D_\alpha\Lambda$ and $\delta B_m = \partial_m\Lambda$ to learn that

$$
\begin{aligned}
U_{\mathrm{U}(5)} &= \lambda^\alpha A_\alpha + e^t(d_a B^a + \partial(s-2t)W^+) + \lim_{y\to z} : e^t\Pi^a(y)A_a(z) : +\\
&\quad + \lim_{y\to z} : \frac{1}{4}e^{t-s}\epsilon^{abcde}u_{ab}d_c d_d(y)A_e(z) : - \lim_{y\to z} : \frac{1}{24}e^{2t-s}\epsilon^{abcde}d_a d_b d_c d_d(y)A_e(z) :
\end{aligned}
\tag{5.4}
$$

where $\lim_{y\to z} : f(y)g(z) := \frac{1}{2\pi i}\oint dy(y-z)^{-1}f(y)g(z)$. Note that up to normal ordering, $U_{\mathrm{U}(5)}$ of (5.4) can be obtained from $Q_{\mathrm{U}(5)}$ of (4.8) by replacing

$$
d_\alpha \to d_\alpha + A_\alpha\,, \qquad \Pi_m \to \Pi_m + B_m\,, \qquad \text{and} \qquad \partial\theta^\alpha \to \partial\theta^\alpha + W^\alpha\,, \tag{5.5}
$$

as was suggested in [7].

As will be shown in subsection 6.2, the massless integrated U(5) vertex operator $V_{\mathrm{U}(5)}$ can be obtained from $U_{\mathrm{U}(5)}$ of (5.4) by

$$
\int V_{\mathrm{U}(5)} = \int [(b_{\mathrm{U}(5)})_{-1}U_{\mathrm{U}(5)} + Q_{\mathrm{U}(5)}((b_{\mathrm{U}(5)})_0 V)] \tag{5.6}
$$

where $(b_{\mathrm{U}(5)})_{-1}$ and $(b_{\mathrm{U}(5)})_0$ are modes of the U(5) b ghost of (6.1) and V is the pure spinor integrated vertex operator of (2.17). One can check that $V_{\mathrm{U}(5)}$ is a "special" U(5) vertex operator satisfying $Q_{\mathrm{U}(5)}V_{\mathrm{U}(5)} = \partial U_{\mathrm{U}(5)}$ whose e^t-independent component is V of (2.17).

By taking suitable OPE's of $V_{\mathrm{U}(5)}$ and $U_{\mathrm{U}(5)}$ of (5.6) and (5.4), one can also construct "special" U(5) vertex operators for massive states. For example, vertex operators with $(\text{mass})^2 = N$ can be obtained by taking the contour integral of $V_{\mathrm{U}(5)}$ around $U_{\mathrm{U}(5)}$ where the momenta k^m and l^m of these two vertex operators are chosen to satisfy $(k+l)^2 = N$. It might be possible to use such a construction to confirm the conjecture of (5.1) for all massive states.

5.4 Equivalence of tree amplitudes

Using the results of the previous subsections, it will now be shown that the RNS and pure spinor tree amplitude prescriptions are equivalent. Using RNS vertex operators in the cohomology of $Q' = Q_{RNS} + \oint \eta$, N-point tree amplitudes can be defined by computing the correlation function

$$A_{RNS} = \int dz_4 \cdots dz_N \left\langle U_1(z_1)\, U_2(z_2)\, U_3(z_3) \int dz_4 V_4(z_4) \cdots \int dz_N V_N(z_N) \right\rangle \quad (5.7)$$

where U_r are dimension-zero vertex operators satisfying $Q'U_r = 0$ and V_r are dimension-one vertex operators satisfying $Q'V_r = \partial U_r$.

Normally, one uses the normalization that $\langle c\partial c\partial^2 c e^{-2\phi}\rangle = 1$, but such a normalization would break gauge invariance since it implies that the amplitude vanishes unless the sum of the pictures of the vertex operators is -2. So it will instead be convenient to use the normalization prescription that

$$\langle c\partial c\partial^2 c e^{-2\phi} + Q'\Lambda\rangle = 1 \quad (5.8)$$

for any gauge parameter Λ. For example, $\langle c\partial c\eta\rangle = \langle c\partial c\partial^2 c e^{-2\phi} + Q'(\xi c\partial c\partial^2 c e^{-2\phi})\rangle = 1$ using this normalization. Note that this normalization prescription is invariant under picture-changing and agrees with the standard prescription when the sum of the pictures of the vertex operators is -2.

To explicitly compute (5.7) with the normalization of (5.8), use the free field OPE's to write $A_{RNS} = \int dz_4 \cdots dz_N \langle f(z_1)\rangle$ where $f(z_1)$ is some operator located at the point z_1. Since the external vertex operators are BRST invariant and $c\partial c\partial^2 c e^{-2\phi}$ is the only non-trivial element in the cohomology of Q' at ghost-number $+3$, $f = c\partial c\partial^2 c e^{-2\phi} F(z_r, k_r, \eta_r) + Q'(\Lambda(z_1))$ for some gauge parameter $\Lambda(z_1)$ and for some function $F(z_r, k_r, \eta_r)$ which depends on the external momenta and polarizations. So $A_{RNS} = \int dz_4 \cdots dz_N F(z_r, k_r, \eta_r)$.

To compare with the pure spinor tree amplitude prescription of subsection 2.3, use the results of the previous subsections to map the RNS vertex operators of (5.7) into the pure spinor vertex operators of (2.18). Under this map, one can check that the RNS operator $c\partial c\partial^2 c e^{-2\phi}$ gets mapped into the pure spinor operator $(\theta\gamma^{mnp}\theta) \times (\lambda\gamma_m\theta)(\lambda\gamma_n\theta)(\lambda\gamma_p\theta)$, which is the unique ghost-number $+3$ element in the cohomology of $Q = \oint \lambda^\alpha d_\alpha$. So the RNS normalization of (5.8) coincides with the normalization prescription of (2.20), implying that the tree amplitude prescriptions are equivalent.

6. Pure spinor b ghost

As mentioned earlier, there is a ghost-number -1 operator in the RNS cohomology, $\int d^2 z\, \mu(z)\, b_{RNS}(z)$ where $\mu(z)$ is a Beltrami differential, which has no super-Poincaré invariant counterpart in the pure spinor cohomology. This operator is in the RNS cohomology because its anticommutator with Q_{RNS} is $\int d^2z\, \mu(z)\, T_{RNS}(z)$, which is a total derivative in the moduli space of Riemann surfaces.

Although there are no super-Poincaré covariant operators of negative ghost number, the conjecture of (5.1) implies that the OPE of this operator with any physical state of positive ghost number can be expressed as a super-Poincaré covariant pure spinor vertex operator. As will now be shown, one can define a pure spinor version of the b ghost which, although not super-Poincaré invariant, can be used to construct pure spinor integrated vertex operators from unintegrated operators, to convert pure spinor string antifields into fields, and to define pure spinor tree amplitudes in a worldsheet reparameterization invariant manner. It is hoped that this operator will also be useful for defining loop amplitudes using the pure spinor formalism.

6.1 Construction of the pure spinor b ghost

To express the b ghost in pure spinor language, it is useful to first remove all negative powers of e^t from the U(5) version of the b_{RNS} ghost of (4.9) by adding the BRST-trivial operator $Q_{\mathrm{U}(5)}(-e^{-s-t})$. With this modification,

$$\begin{aligned} b_{\mathrm{U}(5)} &= e^{-t}d_+ + \frac{1}{2}v^{ab}\partial\theta_{ab} + Q_{\mathrm{U}(5)}(-e^{-s-t}) \\ &= e^{-s}\left(d_a\Pi^a + \partial(s-t)\partial\theta^+ - \frac{1}{4}\partial^2\theta^+\right) + \frac{1}{2}v^{ab}\partial\theta_{ab} + e^{t-2s}(d)^5\,. \end{aligned} \tag{6.1}$$

Although $b_{\mathrm{U}(5)}$ is an ASPC operator, its e^t-independent component is not Lorentz invariant so $b_{\mathrm{U}(5)}$ is not a "special" U(5) vertex operator as defined in (5.2). Nevertheless, it will be useful to define the pure spinor version of the b ghost to be the e^t-independent component of (6.1), i.e.

$$b = e^{-s}\left(d_a\Pi^a + \partial(s-t)\partial\theta^+ - \frac{1}{4}\partial^2\theta^+\right) + \frac{1}{2}v^{ab}\partial\theta_{ab}\,. \tag{6.2}$$

It is interesting to note that b can also be written as $\lim_{y\to z} : e^{-s(y)}b^+(z) :$ where b^+ is the $1_{\frac{5}{2}}$ component of the covariantly transforming spinor

$$b^\alpha = \frac{1}{2}\gamma_m^{\alpha\beta}d_\beta\Pi^m + \frac{1}{4}(\gamma^{mn})^\alpha{}_\beta N_{mn}\partial\theta^\beta - \frac{1}{4}\partial^2\theta^\alpha + \frac{1}{2}\partial h\partial\theta^\alpha \tag{6.3}$$

and $\lim_{y\to z} :\quad:$ is defined as in (5.4). One can check that $\{Q, b^\alpha\} = \lim_{y\to z} : \lambda^\alpha(y)T(z) :$ where T is defined in (4.1), so $\{Q, b\} = T$ as desired.

If $U_{\mathrm{U}(5)}$ is a physical U(5) vertex operator of positive ghost number N, then $(\int \mu\, b_{\mathrm{U}(5)})U_{\mathrm{U}(5)}$ is an ASPC physical vertex operator of ghost number $N-1$. Although $(\int \mu\ b_{\mathrm{U}(5)})U_{\mathrm{U}(5)}$ is not "special" since its e^t-independent component is not a pure spinor vertex operator, the conjecture of (5.1) implies that it is related to a "special" U(5) vertex operator $V_{\mathrm{U}(5)}$ by an ASPC gauge transformation, i.e.

$$V_{\mathrm{U}(5)} = \left(\int \mu\, b_{\mathrm{U}(5)}\right) U_{\mathrm{U}(5)} + Q_{\mathrm{U}(5)}\Lambda_{\mathrm{U}(5)} \tag{6.4}$$

where $\Lambda_{\mathrm{U}(5)}$ is an ASPC operator. Taking the e^t-independent component of (6.4), one learns that for any pure spinor vertex operator U at ghost number N, there exists an ASPC operator Λ such that

$$V = \left(\int \mu\, b\right) U + Q\Lambda \tag{6.5}$$

where V is a physical pure spinor vertex operator of ghost number $N-1$.[14] This property will now be used to relate physical pure spinor vertex operators of different ghost numbers.

6.2 Integrated vertex operators from unintegrated operators

In bosonic and RNS string theory, one can use the b_{-1} mode of the b ghost to construct ghost-number zero integrated vertex operators from ghost-number $+1$ unintegrated operators. The only new feature in the pure spinor formalism is that b is not super-Poincaré invariant, so one needs to perform an ASPC gauge transformation as in (6.5) in order that the resulting integrated operator is super-Poincaré covariant.

It will now be shown that when there exists a super-Poincaré covariant operator V which is a dimension one primary field[15] satisfying $QV = \partial U$, the ASPC gauge parameter Λ satisfying (6.5) can be chosen as $\Lambda = \int b_0 V$. To prove this, note that

$$\int Q(b_0 V) = \int (T_0 V + b_0 \partial U) = \int (V - b_{-1}U + \partial(b_0 U)) = \int (V - b_{-1}U) \tag{6.6}$$

where T_0 is the zero mode of the stress tensor which satisfies $T_0 V = V$ for dimension one primary fields. So $\int V = \int b_{-1}U + Q\Lambda$ as desired.

6.3 Fields and antifields

In bosonic and RNS string theory, the presence of the b zero mode implies that there is a doubling of the cohomology at ghost numbers $+1$ and $+2$ where the ghost number $+1$ states are associated with the string field U and the ghost number $+2$ states are associated with the string antifield U^*. In the gauge where the antifield U^*

[14] Note that the RNS and pure spinor ghost numbers coincide since after including the topological sector, the RNS ghost charge is $\oint(cb + \eta\xi + \frac{1}{2}u_{ab}v^{ab}) = \oint(\partial s + \partial t + \frac{1}{2}u_{ab}v^{ab})$. When acting on pure spinor states (which have no e^t dependence), this ghost charge is equivalent to the pure spinor ghost charge $2\oint \partial h = \oint(-3\partial s + \partial t + \frac{1}{2}u_{ab}v^{ab})$ which was defined in subsection 2.1.

[15] For the dimension one massless vertex operator of (2.17), V is primary when A_α is in Lorentz gauge, i.e. $\gamma_m^{\alpha\beta}\partial^m D_\alpha A_\beta = 0$.

is annihilated by T_0, b_0U^* is a physical ghost number +1 field. This procedure can be applied to the pure spinor formalism with the only new feature being that b_0U^* is only super-Poincaré covariant after performing an ASPC gauge transformation as in (6.5).

For example, for physical massless states, the pure spinor ghost number +2 vertex operator is

$$U^* = (\lambda\gamma^{mnpqr}\lambda)C_{mnpqr}(x,\theta) \tag{6.7}$$

where $C_{mnpqr}(x,\theta)$ is a five-form superfield. U^* is annihilated by Q if

$$\lambda^\alpha(\lambda\gamma^{mnpqr}\lambda)D_\alpha C_{mnpqr} = 0$$

for any pure spinor λ^α, which implies that

$$\gamma^{mnpqr}_{(\alpha\beta}D_{\gamma)}C_{mnpqr} = \gamma^m_{(\alpha\beta}H_{\gamma)m} \tag{6.8}$$

for some $H_{\gamma m}$. Furthermore, $\delta U^* = Q\Omega$ implies the gauge transformation

$$\delta C_{mnpqr} = (\gamma_{mnpqr})^{\alpha\beta}D_\alpha\Lambda_\beta \tag{6.9}$$

where $\Omega = \lambda^\alpha\Lambda_\alpha$. To this author's knowledge, this is the first time that super-Maxwell antifields have been described in $d = 10$ superspace.

To perform a component analysis of C_{mnpqr}, it is convenient to choose a gauge in which the lowest component of C_{mnpqr} is cubic in θ^α. In this gauge,

$$C_{mnpqr} = (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]})^\alpha\psi^*_\alpha(x) + (\theta\gamma_{[mnp}\theta)(\theta\gamma_{qr]s}\theta)a^{*s}(x) + \cdots \tag{6.10}$$

where $\psi^*_\alpha(x)$ is the antifield for the photino $\psi^\alpha(x)$, $a^{*s}(x)$ is the antifield for the photon $a_s(x)$, and all component fields in $\cdots$ can be expressed in terms of ψ^*_α and a^{*s}. These antifields satisfy the equation of motion $\partial_m a^{*m} = 0$ with the gauge transformations

$$\delta a^{*m} = \partial_n(\partial^m\omega^n - \partial^n\omega^m), \quad \delta\psi^*_\alpha = \gamma^m_{\alpha\beta}\partial_m\phi^\beta \tag{6.11}$$

where ω^m and ϕ^β are gauge parameters. As usual, the gauge transformations of the antifields are related to the equations of motion of the corresponding fields. It is interesting to note that, up to proportionality constants,

$$\langle U^*U\rangle = \int d^{10}x(a^{*m}(x)a_m(x) + \psi^*_\alpha(x)\psi^\alpha(x)) \tag{6.12}$$

where U and U^* have been gauge-fixed to the form

$$\begin{aligned} U &= \lambda^\alpha A_\alpha(x,\theta) = (\lambda\gamma^m\theta)a_m(x) + (\lambda\gamma^m\theta)(\theta\gamma_m)_\alpha\psi^\alpha(x) + \cdots, \\ U^* &= (\lambda\gamma^{mnpqr}\lambda)C_{mnpqr}(x,\theta) \\ &= (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mn})^\alpha\psi^*_\alpha(x) + (\lambda\gamma^m\theta)(\lambda\gamma^n\theta)(\theta\gamma_{mnp}\theta)a^{*p}(x) + \cdots, \end{aligned}$$

and $\langle\ \rangle$ is defined using the zero mode prescription of (2.20).

6.4 Reparameterization invariant tree amplitude prescription

In bosonic and RNS string theories, one can compute N-point amplitudes in a worldsheet reparameterization invariant manner by defining

$$A = \left\langle \prod_{s=1}^{N-3} \left(\int d^2 z_s \; \mu_s(z_s) b(z_s) \right) U_1(y_1) \cdots U_N(y_N) \right\rangle \tag{6.13}$$

where U_r are unintegrated ghost-number zero vertex operators. The integrand of (6.13) coincides with the integrand of (5.7) when the $N-3$ Beltrami differentials μ_s are chosen to correspond to $N-3$ of the y_r's, and differs by a BRST-trivial operator for other choices of the Beltrami differentials.

The reparameterization invariant prescription of (6.13) can also be used in the pure spinor formalism, but the normalization prescription of (2.20) cannot be directly applied since the integrand of (6.13) is not manifestly super-Poincaré covariant. To define an appropriate normalization prescription which will be denoted by $\langle \; \rangle_{ASPC}$, use the free-field OPE's of section 2 to write

$$A = \int d^2 z_1 \cdots \int d^2 z_{N-3} \langle f(z_s, \eta_r, k_r) \rangle_{ASPC} \tag{6.14}$$

where $f(z_s, \eta_r, k_r)$ is some ghost-number $+3$ operator constructed from ASPC combinations of the spacetime superfields appearing in the external vertex operators. Since f is BRST invariant, the conjecture of (5.1) implies that

$$f = U + Q\Omega \tag{6.15}$$

where U is a super-Poincaré covariant operator and Ω is some ASPC operator constructed from supersymmetric combinations of the spacetime superfields appearing in the external vertex operators.

The normalization prescription $\langle \; \rangle_{ASPC}$ will be defined such that for any ASPC gauge transformation Ω,

$$\langle U + Q\Omega \rangle_{ASPC} = \langle U \rangle \tag{6.16}$$

where $\langle \; \rangle$ is defined in (2.20). In other words, to apply the normalization prescription of $\langle \; \rangle_{ASPC}$, one has to first remove the non-Lorentz covariant part of the integrand by performing an ASPC gauge transformation. The conjecture of (5.1) implies that this procedure is unambiguous since different choices for the ASPC gauge transformation only change the super-Poincaré covariant part by $U \to U + Q\Lambda$ where Λ is a pure spinor gauge transformation. But as was shown in (2.21), $\langle U + Q\Lambda \rangle = \langle U \rangle$ when U and Λ are super-Poincaré covariant operators.

As an example of a computation using the normalization prescription of (6.16), consider the correlation function

$$\int dz_4 \langle U_1(z_1)\, U_2(z_2)\, U_3(z_3)\, b_{-1} U_4(z_4) \rangle_{ASPC} \tag{6.17}$$

where $U = \lambda^\alpha A_\alpha(z)$. Since b is not Lorentz-covariant, $b_{-1}U$ is not Lorentz-covariant. However, as was shown in (6.6), $\int b_{-1}U = \int(V - Q(b_0 V))$ where V is the manifestly Lorentz-covariant operator defined in (2.17). Since $QU = 0$, one can write (6.17) as

$$\int dz_4 \langle U_1(z_1)\, U_2(z_2)\, U_3(z_3)\, V_4(z_4) + Q(U_1(z_1)\, U_2(z_2)\, U_3(z_3)\, b_0 V_4(z_4))\rangle_{ASPC} = \\ = \langle U_1(z_1)\, U_2(z_2)\, U_3(z_3)\, V_4(z_4)\rangle \qquad (6.18)$$

where the second line of (6.18) is evaluated using the prescription of (2.20).

7. Open problems and speculations

In this paper, the pure spinor formalism for the superstring was related to the RNS formalism by finding a field redefinition and similarity transformation which maps the pure spinor BRST operator into the sum of the RNS BRST operator and η_0 ghost. Although this map can be used to relate vertex operators and tree amplitudes in the two formalisms, there remain at least three open problems which need to be resolved before claiming a proof of equivalence of the two formalisms.

Firstly, because the similarity transformation R of (4.12) involves inverse powers of u_{ab}, it was necessary to conjecture in (5.1) that there exists an ASPC gauge choice for physical vertex operators in which these inverse powers of u_{ab} are absent. Although evidence was presented in support of this conjecture, an explicit proof is lacking. Secondly, the hermiticity definition for RNS operators was not shown to coincide with the rather unusual hermiticity definition for pure spinor operators which was discussed in subsection 2.2. And thirdly, it was not yet shown how to use the ASPC b ghost of (6.2) to define pure spinor loop amplitudes which coincide with the RNS loop amplitude prescription.

An interesting feature of this paper is the important role of ASPC operators, i.e. operators which transform covariantly under all supersymmetry transformations and under U(5) Lorentz transformations. A better description of these operators, perhaps using a harmonic superspace, might help to resolve the above open problems. For example, it might be useful to introduce a second pure spinor variable $\overline{\lambda}^\alpha$ into the pure spinor Hilbert space where $\overline{\lambda}^\alpha$ is defined to be the hermitean conjugate of λ^α. One could then describe ASPC operators as pure spinor operators with non-trivial dependence on $\overline{\lambda}^\alpha$. This would make the formalism resemble the $N = 2$ twistor string considered in [17] and [18] where λ^α and $\overline{\lambda}^\alpha$ are pure spinor twistor-like variables satisfying the condition $\Pi^m = \lambda^\alpha \gamma^m_{\alpha\beta} \overline{\lambda}^\beta$ [19]. It is interesting to note that the fermionic $N = 2$ superconformal generators in the $N = 2$ twistor string formalism are $G^+ = \lambda^\alpha d_\alpha$ and $G^- = \overline{\lambda}^\alpha d_\alpha$. Since $\oint G^+$ can be interpreted as the BRST charge, it might be possible to interpret $\overline{\lambda}^\alpha d_\alpha$ as the ASPC b ghost.

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References

[1] N. Berkovits, *Super-poincare covariant quantization of the superstring*, *J. High Energy Phys.* **04** (2000) 018 [hep-th/0001035].

[2] N. Berkovits, *Cohomology in the pure spinor formalism for the superstring*, *J. High Energy Phys.* **09** (2000) 046 [hep-th/0006003].

[3] N. Berkovits and B.C. Vallilo, *Consistency of super-Poincaré covariant superstring tree amplitudes*, *J. High Energy Phys.* **07** (2000) 015 [hep-th/0004171].

[4] D. Friedan, E.J. Martinec and S.H. Shenker, *Conformal invariance, supersymmetry and string theory*, *Nucl. Phys.* **B 271** (1986) 93.

[5] N. Berkovits and B. Zwiebach, *On the picture dependence of Ramond-Ramond cohomology*, *Nucl. Phys.* **B 523** (1998) 311 [hep-th/9711087].

[6] N. Berkovits, *Quantization of the superstring with manifest* U(5) *super-Poincaré invariance*, *Phys. Lett.* **B 457** (1999) 94 [hep-th/9902099].

[7] W. Siegel, *Classical superstring mechanics*, *Nucl. Phys.* **B 263** (1986) 93.

[8] P.S. Howe, *Pure spinors lines in superspace and ten-dimensional supersymmetric theories*, *Phys. Lett.* **B 258** (1991) 141.

[9] I. Bars, *Free fields and new cosets of current algebras*, *Phys. Lett.* **B 255** (1991) 353.

[10] N. Berkovits and M. Bershadsky, unpublished.

[11] J. Polchinski, private communication.

[12] N. Berkovits, *Super-Poincaré invariant superstring field theory*, *Nucl. Phys.* **B 450** (1995) 90 [hep-th/9503099].

[13] N. Berkovits and C. Vafa, $N = 4$ *topological strings*, *Nucl. Phys.* **B 433** (1995) 123 [hep-th/9407190].

[14] N. Berkovits, *Covariant quantization of the Green-Schwarz superstring in a Calabi-Yau background*, *Nucl. Phys.* **B 431** (1994) 258 [hep-th/9404162].

[15] N. Berkovits, *Quantization of the type-II superstring in a curved six-dimensional background*, *Nucl. Phys.* **B 565** (2000) 333 [hep-th/9908041].

[16] S. Cherkis, private communication.

[17] M. Tonin, *World sheet supersymmetric formulations of Green-Schwarz superstrings*, *Phys. Lett.* **B 266** (1991) 312.

[18] N. Berkovits, *The heterotic Green-Schwarz superstring on an $N = (2,0)$ superworldsheet*, *Nucl. Phys.* **B 379** (1992) 96 [hep-th/9201004].

[19] D.P. Sorokin, V.I. Tkach, D.V. Volkov and A.A. Zheltukhin, *From the superparticle Siegel symmetry to the spinning particle proper time supersymmetry*, *Phys. Lett.* **B 216** (1989) 302.